ZNACAJNE GRANIČNE VRIJEDNOSTI FUNKCIJA

U sledećim zadacima ćemo koristiti poznatu graničnu vrijednost:

$\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1}$ ali i manje "varijacije''

$$\lim_{x \to 0} \frac{\sin ax}{ax} = 1 \quad \text{i} \quad \lim_{x \to 0} \frac{\sin^n ax}{(ax)^n} = 1$$

Zadaci:

1) Odrediti sledeće granične vrijednosti:

a) $\lim_{x \to 0} \frac{\sin 4x}{x}$;

b) $\lim_{x \to 0} \frac{tgx}{x}$;

v) $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$;

g) $\lim_{x \to a} \frac{\sin x - \sin a}{x - a}$;

Rješenja:

a) $\lim_{x \to 0} \frac{\sin 4x}{x}$; (i gore i dolje dodamo 4) $= \lim_{x \to 0} \frac{4 \sin 4x}{4x} = 4 \cdot \boxed{\lim_{x \to 0} \frac{\sin 4x}{4x}} = 4 \cdot 1 = 4$

Ovdje smo upotrebili da je $\lim_{x \to 0} \frac{\sin ax}{ax} = 1$

b) $$\lim_{x \to 0} \frac{tgx}{x} = \lim_{x \to 0} \frac{\frac{\sin x}{\cos x}}{x} = \lim_{x \to 0} \frac{\sin x}{x} \cdot \frac{1}{\cos x} = \lim_{x \to 0} \boxed{\frac{\sin x}{x}} \cdot \frac{1}{\cos x} = \lim_{x \to 0} 1 \cdot \frac{1}{\cos x}$$
$$= 1 \cdot \lim_{x \to 0} \frac{1}{\cos x} = 1 \cdot \frac{1}{\cos 0} = 1 \cdot \frac{1}{1} = 1$$

v) $\lim_{x \to 0} \frac{1 - \cos x}{x^2} =$ iskoristićemo formulu iz trigonometrije: $1 - \cos x = 2 \sin^2 \frac{x}{2}$

$$= \lim_{x \to 0} \frac{2 \sin^2 \frac{x}{2}}{x^2} = (\text{ dodamo } 4) = \lim_{x \to 0} 2 \cdot \frac{\sin^2 \frac{x}{2}}{4 \frac{x^2}{4}} = \frac{2}{4} \cdot \boxed{\lim_{x \to 0} \frac{\sin^2 \frac{x}{2}}{\left(\frac{x}{2}\right)^2}} = \frac{1}{2} \cdot 1 = \frac{1}{2}$$

2) Izračunati sledeće granične vrijednosti:

a) $\lim\limits_{x\to 0}\dfrac{\sin 4x}{\sqrt{x+1}-1}$;

b) $\lim\limits_{x\to 1}\dfrac{\sin(1-x)}{\sqrt{x}-1}$;

a) $\lim\limits_{x\to 0}\dfrac{\sin 4x}{\sqrt{x+1}-1} =$ najprije racionalizacija

$$= \lim_{x\to 0}\frac{\sin 4x}{\sqrt{x+1}-1}\cdot\frac{\sqrt{x+1}+1}{\sqrt{x+1}+1} =$$

$$= \lim_{x\to 0}\frac{\sin 4x\cdot\left(\sqrt{x+1}+1\right)}{x+1-1} = \lim_{x\to 0}\frac{\sin 4x\cdot\left(\sqrt{x+1}+1\right)}{x}$$

sad i gore i dolje dodamo 4

$$= \lim_{x\to 0}\frac{4\sin 4x\left(\sqrt{x+1}+1\right)}{4x} = \lim_{x\to 0}4\boxed{\frac{\sin 4x}{4x}}\left(\sqrt{x+1}+1\right) = \lim_{x\to 0}4\cdot 1\cdot\left(\sqrt{x+1}+1\right) =$$

$$= 4\left(\sqrt{0+1}+1\right) = 4\cdot 2 = 8$$

b)

$\lim\limits_{x\to 1}\dfrac{\sin(1-x)}{\sqrt{x}-1} =$ najprije racionalizacija

$\lim\limits_{x\to 1}\dfrac{\sin(1-x)}{\sqrt{x}-1}\cdot\dfrac{\sqrt{x}+1}{\sqrt{x}+1} = \lim\limits_{x\to 1}\dfrac{\sin(1-x)(\sqrt{x}+1)}{x-1} =$ sada smena $x-1=t,$ kad $x\to 1$ tad $t\to 0$

$$= \lim_{t\to 0}\frac{\sin(-t)\left(\sqrt{t+1}+1\right)}{t} = \lim_{t\to 0}\frac{-\sin(t)\left(\sqrt{t+1}+1\right)}{t} = -\lim_{t\to 0}\boxed{\frac{\sin t}{t}}\left(\sqrt{t+1}+1\right)$$

$$= -\lim_{t\to 0}1\cdot\left(\sqrt{t+1}+1\right) = -(1+1) = -2$$

U sledećim zadacima ćemo koristiti:

$$\boxed{\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}=e} \qquad \text{i} \qquad \lim_{x\to\infty}\left(1+\frac{1}{ax}\right)^{ax}=e$$

Još nam treba i činjenica da je e^x neprekidna funkcija i važi:

$$\boxed{\lim_{x\to a} e^{f(x)} = e^{\lim\limits_{x\to a} f(x)}}$$

3) Odrediti sledeće granične vrijednosti:

a) $\lim\limits_{x\to\infty}\left(1+\frac{3}{x}\right)^{x}$;

b) $\lim\limits_{x\to\infty}\left(\frac{x+1}{x-1}\right)^{x}$;

c) $\lim\limits_{x\to\infty} x(\ln(x+1)-\ln x)$;

Rješenja:

a)

$$\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^{x} = \text{ovdje gde je 3 mora biti 1, pa ćemo 3 ‘spustiti’ ispod x}$$

$$\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^{x} = \lim_{x\to\infty}\left(1+\frac{1}{\frac{x}{3}}\right)^{x} = \text{ sad kod } x \text{ u eksponentu pomnožimo i podijelimo sa } 3$$

$$\lim_{x\to\infty}\left(1+\frac{3}{x}\right)^{x} = \lim_{x\to\infty}\left(1+\frac{1}{\frac{x}{3}}\right)^{x} = = \lim_{x\to\infty}\left(1+\frac{1}{\frac{x}{3}}\right)^{\frac{x}{3}\cdot 3} = \boxed{\lim_{x\to\infty}\left(1+\frac{1}{\frac{x}{3}}\right)^{\frac{x}{3}}}^{3} = e^{3}$$

b)

$$\lim_{x\to\infty}\left(\frac{x+1}{x-1}\right)^{x} = \qquad \text{u zagradi ćemo dodati 1 i oduzeti 1 } =$$

$$\lim_{x\to\infty}\left(1+\frac{x+1}{x-1}-1\right)^{x} = \lim_{x\to\infty}\left(1+\frac{x+1-1(x-1)}{x-1}\right)^{x} = \lim_{x\to\infty}\left(1+\frac{x+1-x+1}{x-1}\right)^{x}$$

$$= \lim_{x\to\infty}\left(1+\frac{2}{x-1}\right)^{x} = \lim_{x\to\infty}\left(1+\frac{1}{\frac{x-1}{2}}\right)^{x} = \lim_{x\to\infty}\left(1+\frac{1}{\frac{x-1}{2}}\right)^{\frac{x-1}{2}\cdot\frac{2}{x-1}\cdot x} =$$

$$= \lim_{x\to\infty}\left(1+\frac{1}{\frac{x-1}{2}}\right)^{\frac{x-1}{2}\cdot\frac{2x}{x-1}} = \boxed{\lim_{x\to\infty}\left(1+\frac{1}{\frac{x-1}{2}}\right)^{\frac{x-1}{2}}}^{\frac{2x}{x-1}} = \lim_{x\to\infty} e^{\frac{2x}{x-1}} = e^{\lim_{x\to\infty}\frac{2x}{x-1}} = e^{2}$$

v)

$$\lim_{x\to\infty} x\cdot(\ln(x+1)-\ln x) = \lim_{x\to\infty}[x\cdot\ln\frac{x+1}{x}] = \lim_{x\to\infty}\ln\left(\frac{x+1}{x}\right)^{x} =$$

(pošto je ln neprekidna funkcija i ona može da zamijeni mjesto sa lim)

$$\ln\lim_{x\to\infty}\left(\frac{x+1}{x}\right)^{x} = \ln\lim_{x\to\infty}\left(\frac{x}{x}+\frac{1}{x}\right)^{x} = \ln\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x} = \ln e = 1$$

Ovde smo koristili pravila(pogledaj II godina logaritmi): $\ln A - \ln B = \ln\frac{A}{B}$ i

$n\cdot\ln A = \ln A^{n}$